# Logarithms*

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## 1 Introduction

In mathematics many ideas are related. We saw that addition and subtraction are related and that multiplication and division are related. Similarly, exponentials and logarithms are related.

Logarithms are commonly refered to as logs, are the "opposite" of exponentials, just as subtraction is the opposite of addition and division is the opposite of multiplication. Logs "undo" exponentials. Technically speaking, logs are the inverses of exponentials. The logarithm of a number $x$ in the base $a$ is defined as the number $n$ such that $a^n = x$.

So, if $a^n = x$, then:

$$log_a(x) = n \tag{1}$$

> ASIDE: When we say "inverse function" we mean that the answer becomes the question and the question becomes the answer. For example, in the equation $a^b = x$ the "question" is "what is $a$ raised to the power $b$?" The answer is "$x$." The inverse function would be $log_a x = b$ or "by what power must we raise $a$ to obtain $x$?" The answer is "$b$."

The mathematical symbol for logarithm is $log_a(x)$ and it is read "log to the base $a$ of $x$". For example, $log_{10}(100)$ is "log to the base 10 of 100."

### 1.1 Logarithm Symbols :

Write the following out in words. The first one is done for you.

1. $log_2(4)$ is log to the base 2 of 4
2. $log_{10}(14)$
3. $log_{16}(4)$
4. $log_x(8)$
5. $log_y(x)$

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## 2 Definition of Logarithms

The logarithm of a number is the value to which the base must be raised to give that number i.e. the exponent. From the first example of the activity $log_2(4)$ means the power of 2 that will give 4. As $2^2 = 4$, we see that

$$log_2(4) = 2 \tag{2}$$

The *exponential-form* is then $2^2 = 4$ and the *logarithmic-form* is $log_2 4 = 2$.

**Definition 1: Logarithms**
If $a^n = x$, then: $log_a(x) = n$, where $a > 0$; $a \neq 1$ and $x > 0$.

### 2.1 Applying the definition :

Find the value of:

1. $log_7 343$

$$\begin{aligned} &\text{Reasoning :} \\ &7^3 = 343 \\ &\text{therefore, } log_7 343 = 3 \end{aligned} \tag{3}$$

2. $log_2 8$
3. $log_4 \frac{1}{64}$
4. $log_{10} 1\ 000$

## 3 Logarithm Bases

Logarithms, like exponentials, also have a base and $log_2(2)$ is not the same as $log_{10}(2)$.

We generally use the "common" base, 10, or the *natural* base, $e$.

The number $e$ is an irrational number between 2.71 and 2.72. It comes up surprisingly often in Mathematics, but for now suffice it to say that it is one of the two common bases.

### 3.1 Natural Logarithm

The natural logarithm (symbol $ln$) is widely used in the sciences. The natural logarithm is to the base $e$ which is approximately 2.71828183.... $e$ is like $\pi$ and is another example of an irrational number.

While the notation $log_{10}(x)$ and $log_e(x)$ may be used, $log_{10}(x)$ is often written $log(x)$ in Science and $log_e(x)$ is normally written as $ln(x)$ in both Science and Mathematics. So, if you see the $log$ symbol without a base, it means $log_{10}$.

It is often necessary or convenient to convert a log from one base to another. An engineer might need an approximate solution to a log in a base for which he does not have a table or calculator function, or it may be algebraically convenient to have two logs in the same base.

Logarithms can be changed from one base to another, by using the change of base formula:

$$log_a x = \frac{log_b x}{log_b a} \tag{4}$$

where $b$ is any base you find convenient. Normally $a$ and $b$ are known, therefore $log_b a$ is normally a known, if irrational, number.

For example, change $log_2 12$ in base 10 is:

$$log_2 12 = \frac{log_{10} 12}{log_{10} 2} \tag{5}$$

### 3.2 Change of Base : Change the following to the indicated base:

1. $log_2(4)$ to base 8
2. $log_{10}(14)$ to base 2
3. $log_{16}(4)$ to base 10
4. $log_x(8)$ to base $y$
5. $log_y(x)$ to base $x$

**Khan academy video on logarithms - 1**

This media object is a Flash object. Please view or download it at
<http://www.youtube.com/v/mQTWzLpCcW0&rel=0&hl=en_US&feature=player_embedded&version=3>

**Figure 1**

## 4 Laws of Logarithms

Just as for the exponents, logarithms have some laws which make working with them easier. These laws are based on the exponential laws and are summarised first and then explained in detail.

$$\begin{array}{rcl} log_a(1) & = & 0 \\ log_a(a) & = & 1 \\ log_a(x \cdot y) & = & log_a(x) + log_a(y) \\ log_a\left(\frac{x}{y}\right) & = & log_a(x) - log_a(y) \\ log_a\left(x^b\right) & = & b log_a(x) \\ log_a\left(\sqrt[b]{x}\right) & = & \frac{log_a(x)}{b} \end{array} \tag{6}$$

## 5 Logarithm Law 1: $log_a 1 = 0$

$$\begin{array}{rcl} \text{Since} \quad a^0 & = & 1 \\ \text{Then,} \quad log_a(1) & = & log_a\left(a^0\right) \\ & = & 0 \quad \text{by definition of logarithm} \end{array} \tag{7}$$

For example,

$$log_2 1 = 0 \tag{8}$$

and

$$log_{25} 1 = 0 \tag{9}$$

### 5.1 Logarithm Law 1: $log_a 1 = 0$ :

Simplify the following:

1. $log_2(1) + 5$
2. $log_{10}(1) \times 100$
3. $3 \times log_{16}(1)$
4. $log_x(1) + 2xy$
5. $\frac{log_y(1)}{x}$

## 6 Logarithm Law 2: $log_a(a) = 1$

$$\begin{aligned} \text{Since} \quad a^1 &= a \\ \text{Then,} \quad log_a(a) &= log_a\left(a^1\right) \\ &= 1 \quad \text{by definition of logarithm} \end{aligned} \tag{10}$$

For example,

$$log_2 2 = 1 \tag{11}$$

and

$$log_{25} 25 = 1 \tag{12}$$

### 6.1 Logarithm Law 2: $log_a(a) = 1$ :

Simplify the following:

1. $log_2(2) + 5$
2. $log_{10}(10) \times 100$
3. $3 \times log_{16}(16)$
4. $log_x(x) + 2xy$
5. $\frac{log_y(y)}{x}$

TIP: Useful to know and remember

*When the base is 10, we do not need to state it.* From the work done up to now, it is also useful to summarise the following facts:

1. $log1 = 0$
2. $log10 = 1$
3. $log100 = 2$
4. $log1000 = 3$

## 7 Logarithm Law 3: $log_a(x \cdot y) = log_a(x) + log_a(y)$

The derivation of this law is a bit trickier than the first two. Firstly, we need to relate $x$ and $y$ to the base $a$. So, assume that $x = a^m$ and $y = a^n$. Then from Equation (1), we have that:

$$\begin{array}{rcl} log_a(x) & = & m \\ \text{and} \quad log_a(y) & = & n \end{array} \tag{13}$$

This means that we can write:

$$\begin{array}{rcl} log_a(x \cdot y) & = & log_a(a^m \cdot a^n) \\ & = & log_a\left(a^{m+n}\right) \quad \text{Exponential laws} \\ & = & log_a\left(a^{log_a(x)+log_a(y)}\right) \\ & = & log_a(x) + log_a(y) \end{array} \tag{14}$$

For example, show that $log(10 \cdot 100) = log10 + log100$. Start with calculating the left hand side:

$$\begin{array}{rcl} log(10 \cdot 100) & = & log(1000) \\ & = & log\left(10^3\right) \\ & = & 3 \end{array} \tag{15}$$

The right hand side:

$$\begin{array}{rcl} log10 + log100 & = & 1 + 2 \\ & = & 3 \end{array} \tag{16}$$

Both sides are equal. Therefore, $log(10 \cdot 100) = log10 + log100$.

### 7.1 Logarithm Law 3: $log_a(x \cdot y) = log_a(x) + log_a(y)$ :

Write as seperate logs:

1. $log_2(8 \times 4)$
2. $log_8(10 \times 10)$
3. $log_{16}(xy)$
4. $log_z(2xy)$
5. $log_x\left(y^2\right)$

## 8 Logarithm Law 4: $log_a\left(\frac{x}{y}\right) = log_a(x) - log_a(y)$

The derivation of this law is identical to the derivation of Logarithm Law 3 and is left as an exercise.

For example, show that $log\left(\frac{10}{100}\right) = log10 - log100$. Start with calculating the left hand side:

$$\begin{array}{rcl} log\left(\frac{10}{100}\right) & = & log\left(\frac{1}{10}\right) \\ & = & log\left(10^{-1}\right) \\ & = & -1 \end{array} \tag{17}$$

The right hand side:

$$\begin{aligned} log10 - log100 &= 1 - 2 \\ &= -1 \end{aligned} \tag{18}$$

Both sides are equal. Therefore, $log\left(\frac{10}{100}\right) = log10 - log100$.

### 8.1 Logarithm Law 4: $log_a\left(\frac{x}{y}\right) = log_a(x) - log_a(y)$ :

Write as seperate logs:

1. $log_2\left(\frac{8}{5}\right)$
2. $log_8\left(\frac{100}{3}\right)$
3. $log_{16}\left(\frac{x}{y}\right)$
4. $log_z\left(\frac{2}{y}\right)$
5. $log_x\left(\frac{y}{2}\right)$

## 9 Logarithm Law 5: $log_a\left(x^b\right) = blog_a(x)$

Once again, we need to relate $x$ to the base $a$. So, we let $x = a^m$. Then,

$$\begin{array}{rrcll} & log_a\left(x^b\right) & = & log_a\left(\left(a^m\right)^b\right) & \\ & & = & log_a\left(a^{m \cdot b}\right) & \text{(exponential laws} \\ \text{But,} & m & = & log_a(x) & \text{(Assumption that } x = a^m) \\ \therefore & log_a\left(x^b\right) & = & log_a\left(a^{b \cdot log_a(x)}\right) & \\ & & = & b \cdot log_a(x) & \text{(Definition of logarithm)} \end{array} \tag{19}$$

For example, we can show that $log_2\left(5^3\right) = 3log_2(5)$.

$$\begin{array}{rcll} log_2\left(5^3\right) & = & log_2(5 \cdot 5 \cdot 5) & \\ & = & log_2 5 + log_2 5 + log_2 5 & \left(\because log_a(x \cdot y) = log_a\left(a^m \cdot a^n\right)\right) \\ & = & 3log_2 5 & \end{array} \tag{20}$$

Therefore, $log_2\left(5^3\right) = 3log_2(5)$.

### 9.1 Logarithm Law 5: $log_a\left(x^b\right) = blog_a(x)$ :

Simplify the following:

1. $log_2\left(8^4\right)$
2. $log_8\left(10^{10}\right)$
3. $log_{16}\left(x^y\right)$
4. $log_z\left(y^x\right)$
5. $log_x\left(y^{2x}\right)$

## 10 Logarithm Law 6: $log_a\left(\sqrt[b]{x}\right) = \frac{log_a(x)}{b}$

The derivation of this law is identical to the derivation of Logarithm Law 5 and is left as an exercise.

For example, we can show that $log_2\left(\sqrt[3]{5}\right) = \frac{log_2 5}{3}$.

$$\begin{array}{rcl} log_2\left(\sqrt[3]{5}\right) & = & log_2\left(5^{\frac{1}{3}}\right) \\ & = & \frac{1}{3}log_2 5 \quad \left(\because log_a\left(x^b\right) = b log_a(x)\right) \\ & = & \frac{log_2 5}{3} \end{array} \quad (21)$$

Therefore, $log_2\left(\sqrt[3]{5}\right) = \frac{log_2 5}{3}$.

### 10.1 Logarithm Law 6: $log_a\left(\sqrt[b]{x}\right) = \frac{log_a(x)}{b}$ :

Simplify the following:

1. $log_2\left(\sqrt[4]{8}\right)$
2. $log_8\left(\sqrt[10]{10}\right)$
3. $log_{16}\left(\sqrt[y]{x}\right)$
4. $log_z\left(\sqrt[x]{y}\right)$
5. $log_x\left(\sqrt[2x]{y}\right)$

TIP: The final answer doesn't have to *look* simple.

**Khan academy video on logarithms - 2**

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<http://www.youtube.com/v/PupNgv49_WY&rel=0&hl=en_US&feature=player_embedded&version=3>

**Figure 2**

**Khan academy video on logarithms - 3**

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<http://www.youtube.com/v/TMmxKZaCqe0&rel=0&hl=en_US&feature=player_embedded&version=3>

**Figure 3**

**Exercise 1: Simplification of Logs** *(Solution on p. 11.)*

Simplify, without use of a calculator:

$$3log3 + log125 \quad (22)$$

**Exercise 2: Simplification of Logs** *(Solution on p. 11.)*

Simplify, without use of a calculator:

$$8^{\frac{2}{3}} + log_2 32 \qquad (23)$$

**Exercise 3: Simplify to one log** *(Solution on p. 11.)*

Write $2log3 + log2 - log5$ as the logarithm of a single number.

TIP: Exponent rule: $\left(x^b\right)^a = x^{ab}$

## 11 Solving simple log equations

In grade 10 you solved some exponential equations by trial and error, because you did not know the great power of logarithms yet. Now it is much easier to solve these equations by using logarithms.

For example to solve $x$ in $25^x = 50$ correct to two decimal places you simply apply the following reasoning. If the LHS = RHS then the logarithm of the LHS must be equal to the logarithm of the RHS. By applying Law 5, you will be able to use your calculator to solve for $x$.

**Exercise 4: Solving Log equations** *(Solution on p. 11.)*

Solve for $x$: $25^x = 50$ correct to two decimal places.

In general, the exponential equation should be simplified as much as possible. Then the aim is to make the unknown quantity (i.e. $x$) the subject of the equation.

For example, the equation

$$2^{(x+2)} = 1 \qquad (24)$$

is solved by moving all terms with the unknown to one side of the equation and taking all constants to the other side of the equation

$$\begin{array}{rcl} 2^x \cdot 2^2 & = & 1 \\ 2^x & = & \frac{1}{2^2} \end{array} \qquad (25)$$

Then, take the logarithm of each side.

$$\begin{array}{rcc} log\left(2^x\right) & = & log\left(\frac{1}{2^2}\right) \\ xlog\left(2\right) & = & -log\left(2^2\right) \\ xlog\left(2\right) & = & -2log\left(2\right) \quad \text{Divide both sides by } log\left(2\right) \\ \therefore \quad x & = & -2 \end{array} \qquad (26)$$

Substituting into the original equation, yields

$$2^{-2+2} = 2^0 = 1 \qquad (27)$$

Similarly, $9^{(1-2x)} = 3^4$ is solved as follows:

$$\begin{array}{rcl} 9^{(1-2x)} & = & 3^4 \\ 3^{2(1-2x)} & = & 3^4 \\ 3^{2-4x} & = & 3^4 \quad \text{take the logarithm of both sides} \\ log\left(3^{2-4x}\right) & = & log\left(3^4\right) \\ (2-4x)\,log\,(3) & = & 4log\,(3) \quad \text{divide both sides by } log\,(3) \\ 2-4x & = & 4 \\ -4x & = & 2 \\ \therefore x & = & -\frac{1}{2} \end{array} \quad (28)$$

Substituting into the original equation, yields

$$9^{\left(1-2\left(\frac{-1}{2}\right)\right)} = 9^{(1+1)} = 3^{2(2)} = 3^4 \quad (29)$$

**Exercise 5: Exponential Equation** ***(Solution on p. 11.)***
Solve for $x$ in $7 \cdot 5^{(3x+3)} = 35$

### 11.1 Exercises

Solve for $x$:

1. $log_3 x = 2$
2. $10^{log27} = x$
3. $3^{2x-1} = 27^{2x-1}$

## 12 Logarithmic applications in the Real World

Logarithms are part of a number of formulae used in the Physical Sciences. There are formulae that deal with earthquakes, with sound, and pH-levels to mention a few. To work out time periods is growth or decay, logs are used to solve the particular equation.

**Exercise 6: Using the growth formula** ***(Solution on p. 12.)***
A city grows 5% every 2 years. How long will it take for the city to triple its size?

**Exercise 7: Logs in Compound Interest** ***(Solution on p. 12.)***
I have R12 000 to invest. I need the money to grow to at least R30 000. If it is invested at a compound interest rate of 13% per annum, for how long (in full years) does my investment need to grow ?

### 12.1 Exercises

1. The population of a certain bacteria is expected to grow exponentially at a rate of 15 % every hour. If the initial population is 5 000, how long will it take for the population to reach 100 000 ?
2. Plus Bank is offering a savings account with an interest rate if 10 % per annum compounded monthly. You can afford to save R 300 per month. How long will it take you to save R 20 000 ? (Give your answer in years and months)

## 13 End of Chapter Exercises

1. Show that
$$log_a\left(\frac{x}{y}\right) = log_a(x) - log_a(y) \qquad (30)$$
2. Show that
$$log_a\left(\sqrt[b]{x}\right) = \frac{log_a(x)}{b} \qquad (31)$$
3. Without using a calculator show that:
$$log\frac{75}{16} - 2log\frac{5}{9} + log\frac{32}{243} = log2 \qquad (32)$$
4. Given that $5^n = x$ and $n = log_2 y$
   a. Write $y$ in terms of $n$
   b. Express $log_8 4y$ in terms of $n$
   c. Express $50^{n+1}$ in terms of $x$ and $y$
5. Simplify, without the use of a calculator:
   a. $8^{\frac{2}{3}} + log_2 32$
   b. $log_3 9 - log_5 \sqrt{5}$
   c. $\left(\frac{5}{4^{-1}-9^{-1}}\right)^{\frac{1}{2}} + log_3 9^{2,12}$
6. Simplify to a single number, without use of a calculator:
   a. $log_5 125 + \frac{log32 - log8}{log8}$
   b. $log3 - log0,3$
7. Given: $log_3 6 = a$ and $log_6 5 = b$
   a. Express $log_3 2$ in terms of $a$.
   b. Hence, or otherwise, find $log_3 10$ in terms of $a$ and $b$.
8. Given: $pq^k = qp^{-1}$ Prove: $k = 1 - 2log_q p$
9. Evaluate without using a calculator: $(log_7 49)^5 + log_5\left(\frac{1}{125}\right) - 13\,log_9 1$
10. If $log5 = 0,7$, determine, **without using a calculator**:
    a. $log_2 5$
    b. $10^{-1,4}$
11. Given: $M = log_2(x+3) + log_2(x-3)$
    a. Determine the values of $x$ for which $M$ is defined.
    b. Solve for $x$ if $M = 4$.
12. Solve: $\left(x^3\right)^{logx} = 10x^2$ (Answer(s) may be left in surd form, if necessary.)
13. Find the value of $(log_{27} 3)^3$ without the use of a calculator.
14. Simplify By using a calculator: $log_4 8 + 2log_3\sqrt{27}$
15. Write $log4500$ in terms of $a$ and $b$ if $2 = 10^a$ and $9 = 10^b$.
16. Calculate: $\frac{5^{2006} - 5^{2004} + 24}{5^{2004} + 1}$
17. Solve the following equation for $x$ without the use of a calculator and using the fact that $\sqrt{10} \approx 3,16$ :
$$2log(x+1) = \frac{6}{log(x+1)} - 1 \qquad (33)$$
18. Solve the following equation for $x$: $6^{6x} = 66$ (Give answer correct to 2 decimal places.)

# Solutions to Exercises in this Module

**Solution to Exercise (p. 7)**

Step 1. 125 can be written as $5^3$.
Step 2.

$$\begin{array}{rcl} 3log3+log125 & = & 3log3+log5^3 \\ & = & 3log3+3log5 \quad \because log_a\left(x^b\right)=blog_a\left(x\right) \end{array} \quad (34)$$

Step 3. We cannot simplify any further. The final answer is:

$$3log3+3log5 \quad (35)$$

**Solution to Exercise (p. 7)**

Step 1. 8 can be written as $2^3$. 32 can be written as $2^5$.
Step 2.

$$8^{\frac{2}{3}}+log_2 32=\left(2^3\right)^{\frac{2}{3}}+log_2 2^5 \quad (36)$$

Step 3. We can use:

$$log_a\left(x^b\right)=blog_a\left(x\right) \quad (37)$$

Step 4.

$$\left(2^3\right)^{\frac{2}{3}}+log_2 2^5=\left(2\right)^{3\frac{2}{3}}+5log_2 2 \quad (38)$$

Step 5. We can now use $log_a a=1$
Step 6.

$$\left(2\right)^{3\frac{2}{3}}+5log_2 2=\left(2\right)^2+5\left(1\right)=4+5=9 \quad (39)$$

Step 7. The final answer is:

$$8^{\frac{2}{3}}+log_2 32=9 \quad (40)$$

**Solution to Exercise (p. 8)**

Step 1. $2log3+log2-log5=log3^2+log2-log5$
Step 2. $=log\left(3^2\times 2\div 5\right)$
Step 3. $=log3,6$

**Solution to Exercise (p. 8)**

Step 1. $log25^x=log50$
Step 2. $xlog25=log50$
Step 3. $x=log50\div log25$
$x=1,21533....$
Step 4. $x=1,22$

**Solution to Exercise (p. 9)**

Step 1. There are two possible bases: 5 and 7. $x$ is an exponent of 5.
Step 2. In order to eliminate 7, divide both sides of the equation by 7 to give:

$$5^{(3x+3)}=5 \quad (41)$$

Step 3.

$$log\left(5^{(3x+3)}\right) = log\,(5) \tag{42}$$

Step 4.

$$\begin{array}{rcl} (3x+3)\,log\,(5) & = & log\,(5) \quad \text{divide both sides of the equation by } log\,(5) \\ 3x+3 & = & 1 \\ 3x & = & -2 \\ x & = & -\frac{2}{3} \end{array} \tag{43}$$

Step 5.

$$7\cdot 5^{\left(-3\frac{2}{3}+3\right)} = 7\cdot 5^{(-2+3)} = 7\cdot 5^{1} = 35 \tag{44}$$

**Solution to Exercise (p. 9)**

Step 1. $A = P(1+i)^n$ Assume $P = x$, then $A = 3x$. For this example $n$ represents a period of 2 years, therefore the $n$ is halved for this purpose.

Step 2.

$$\begin{array}{rcl} 3 & = & (1,05)^{\frac{n}{2}} \\ log3 & = & \frac{n}{2}\times log1,05 \quad \text{(using law 5)} \\ n & = & 2log3 \div log1,05 \\ n & = & 45,034 \end{array} \tag{45}$$

Step 3. So it will take approximately 45 years for the population to triple in size.

**Solution to Exercise (p. 9)**

Step 1.

$$A = P(1+i)^n \tag{46}$$

Step 2.

$$\begin{array}{rcl} 30000 & < & 12000(1+0,13)^n \\ 1,13^n & > & \frac{5}{2} \\ nlog\,(1,13) & > & log\,(2,5) \\ n & > & log\,(2,5) \div log\,(1,13) \\ n & > & 7,4972... \end{array} \tag{47}$$

Step 3. In this case we round up, because 7 years will not yet deliver the required R 30 000. The investment need to stay in the bank for at least 8 years.